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FACTORING FUNCTIONS OF MAUREY'S FACTORIZATION THEOREM

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ABSTRACT
It is proved that the best possible factoring functions of Maurey's factorization theorem are unique up to multiplication by constant-modular functions.

1. Introduction

Let $L^s(\Omega,\mu)$ denote a usual L^s-space on an arbitrary measure space $\{\Omega,\mu\}$. The norm or quasi-norm of $L^s(\Omega,\mu)$ is denoted by $\|f\|_s$. Our argument on Maurey's factorization theorem depends on the following facts (Lemma 1). Let T denote a nonempty bounded closed convex subset of $L^s(\Omega,\mu)$, each element of which is nonnegative. Then, in the case $0<s<1$, T has a unique function g whose quasi-norm is maximal in T. This extremal function g satisfies the inequality

$$\int (f-g)g^{s-1}d\mu \leqq 0, \qquad \forall f \in T.$$

In the case $1<s<+\infty$, the above inequality is reversed, that is, in any T of $L^s(\Omega,\mu)$ as cited above there exists a unique function g which satisfies the inequality

$$\int (f-g)g^{s-1}d\mu \geqq 0, \qquad \forall f \in T.$$

From these facts we can directly derive Maurey's factorization theorem. Let u be a continuous linear map from a quasi-normed space E into $L^p(\Omega,\mu)$. This theorem gives certain equivalent conditions for u to be factored as

$$E \xrightarrow{v} L^q(\Omega,\mu) \xrightarrow{(h)} L^p(\Omega,\mu) \qquad (0<p<q),$$

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where (h) is the multiplication operator by a function $h \in L^r(\Omega, \mu)$ and $1/r = 1/p - 1/q$. The theorem also covers the case that a linear map u from a linear subspace S_q of $L^q(\Omega, \mu)$ into E admits a factorization

$$S_q \xrightarrow{(h)} S_p \xrightarrow{v} E.$$

Incidentally our argument yields that the best possible factoring functions h are essentially unique: i.e., the h's are uniquely determined up to multiplication by constant-modular functions, if they minimize the values $\|v\| \cdot \|h\|_r$ among all possible factorizations.

Finally the author expresses his deep gratitude to the referee for his valuable advice.

2. Proof of the statements

We start by recalling Minkowski's inequality: with $f, g \in L^s(\Omega, \mu)$,

$$\left(\int |f|^s d\mu\right)^{1/s} + \left(\int |g|^s d\mu\right)^{1/s} \leqq \left\{\int (|f| + |g|)^s d\mu\right\}^{1/s}, \qquad 0 < s < 1.$$

The equality holds if and only if $|f|/|g| = \text{const. a.e. } \mu$. Of course the converse inequality is valid in the case $1 < s < +\infty$.

The latter half of the following lemma is due to the referee.

Lemma 1. *Let T be a nonempty bounded closed convex subset of $L^s(\Omega, \mu)$. Suppose all functions of T are nonnegative. Then, in the case $0 < s < 1$, T has a unique function g whose quasi-norm is maximum in T. The extremal function g satisfies the inequality*

$$\int (f - g) g^{s-1} d\mu \leqq 0, \qquad \forall f \in T.$$

If $1 < s < +\infty$, T admits a unique function g whose norm is minimum in T and the function g satisfies

$$\int (f - g) g^{s-1} d\mu \geqq 0, \qquad \forall f \in T.$$

In this case the assumption that T is bounded is unnecessary.

Proof. We may assume that $1 = \sup\{\|f\|_s : f \in T\}$. We first deal with the case $1 < s < +\infty$. Since $L^s(\Omega, \mu)$ is reflexive in this case, T is weakly compact. Applying the lower semi-continuity of the norm in the weak topology, we have that T has a function g whose norm is minimum in T. The uniqueness of such g

is due to Minkowski's inequality. Now, assume that a function $f \in T$ satisfies the inequality $\int (f-g)g^{s-1}d\mu < 0$. It is easy to check, using the dominated convergence theorem, that

$$\frac{d}{d\varepsilon}\int [\varepsilon f+(1-\varepsilon)g]^s d\mu = s\int [\varepsilon f+(1-\varepsilon)g]^{s-1}(f-g)d\mu$$

so

$$\frac{d}{d\varepsilon}\int [\varepsilon f+(1-\varepsilon)g]^s d\mu \Big|_{\varepsilon=0} = s\int g^{s-1}(f-g)d\mu < 0.$$

Hence for $0<\varepsilon$ small enough we get $\|\varepsilon f+(1-\varepsilon)g\|_s^s < \|g\|_s^s$, a contradiction to the minimality of $\|g\|_s$. This establishes the latter half of the assertion.

In the sequel we suppose that $0<s<1$. Pick up an arbitrary sequence $\{g_n\}$ from T so that $\|g_n\|_s \to 1 = \sup\{\|f\|_s : f \in T\}$. We will prove that such a sequence is always convergent in L^s, which will lead to the existence and uniqueness of the desired function. We first show that the sequence $\{(g_n)^s\}$ in L^1 converges in norm. Assume that this is false. Then since $\int (g_n)^s d\mu \to 1$, there exist two subsequences $\{m_k\}$ and $\{n_k\}$ of $\{n\}$ and a positive number c such that $n_k < m_k$ and

$$\int_{E_k} [(g_{m_k})^s - (g_{n_k})^s]d\mu \geqq c > 0, \qquad k \in \mathbf{N},$$

where

$$E_k = \Omega[g_{n_k} < g_{m_k}] = \{\omega \in \Omega : g_{n_k}(\omega) < g_{m_k}(\omega)\}.$$

For each E_k we define a quasi-norm preserving map $f \to [f]_k = (f_1, f_2)$ from T into l_2^s by the relation

$$f_1 = \left(\int_{E_k} f^s d\mu\right)^{1/s}, \qquad f_2 = \left(\int_{\Omega\setminus E_k} f^s d\mu\right)^{1/s}.$$

By Minkowski's inequality, we have that for any pair $f, g \in T$

$$\begin{aligned} qf_1 + pg_1 &= \left(\int_{E_k} (qf)^s d\mu\right)^{1/s} + \left(\int_{E_k} (pg)^s d\mu\right)^{1/s} \\ &\leqq \left(\int_{E_k} (qf+pg)^s d\mu\right)^{1/s} = (qf+pg)_1 \end{aligned}$$

with the constants p, q nonnegative. Similarly the inequality $qf_2 + pg_2 \leqq (qf+pg)_2$ holds. These yield that

$$\|q[g_{n_k}]_k + p[g_{m_k}]_k\|_s \leqq \|[qg_{n_k} + pg_{m_k}]_k\|_s = \|qg_{n_k} + pg_{m_k}\|_s.$$

Therefore each convex combination of $[g_{n_k}]_k$ and $[g_{m_k}]_k$ has the quasi-norm bounded by 1. The same is true for two vectors $\alpha = \lim_k [g_{m_k}]_k$, $\beta = \lim_k [g_{n_k}]_k$, if they exist. Surely we can suppose so, by considering subsequences of $\{n_k\}$ and $\{m_k\}$ if necessary. Then the assumption

$$\int_{E_k} [(g_{m_k})^s - (g_{n_k})^s] d\mu \geqq c > 0$$

implies $\alpha \neq \beta$, while $\|\alpha\|_s = \|\beta\|_s = 1$ by $\int (g_n)^s d\mu \to 1$. This is evidently a contradiction, because the implicit function $y = (1 - x^s)^{1/s}$, $0 \leqq x \leqq 1$, of $x^s + y^s = 1$ is strictly convex. So $\{(g_n)^s\}$ converges to some function G of L^1 in norm. Clearly $G \geqq 0$, $\int G\, d\mu = 1$. Here putting $g = G^{1/s}$, we claim that $g_n \to g$ in the s-th mean. Take any number ε from the open interval $(0,1)$ and call

$$A_\varepsilon = \sup \left\{ \frac{(1-x)^s}{1-x^s} : 0 \leqq x \leqq 1 - \varepsilon \right\}.$$

It is easy to see that on a set $W_n = \Omega[(1-\varepsilon)g \leqq g_n \leqq (1-\varepsilon)^{-1} g]$ the inequality $|g - g_n|^s \leqq (\varepsilon/1-\varepsilon)^s g^s$ holds, and on $\Omega \setminus W_n$ we have $|g - g_n|^s \leqq A_\varepsilon |g^s - g_n^s|$. Therefore

$$\int |g - g_n|^s d\mu = \int_{W_n} |g - g_n|^s d\mu + \int_{\Omega \setminus W_n} |g - g_n|^s d\mu$$

$$\leqq (\varepsilon/1-\varepsilon)^s \int g^s d\mu + A_\varepsilon \int |g^s - g_n^s|\, d\mu.$$

This yields $\overline{\lim}_n \int |g - g_n|^s d\mu \leqq (\varepsilon/(1-\varepsilon))^s$. Since ε is arbitrary, we conclude that $g_n \to g$ in the s-th mean. In particular the desired function g exists in T and is also unique.

Finally we show the inequality satisfied by this extremal function: $\int (f - g) g^{s-1} d\mu \leqq 0$, $\forall f \in T$ (conventions $0/0 = 0$). We first note that this integral is definite in Lebesgue's sense, because the negative portion $\int_{\Omega[f \leqq g]} (f - g) g^{s-1} d\mu$ is finite. Assume that a function $f \in T$ satisfies the reverse inequality $\int (f - g) g^{s-1} d\mu > 0$. We decompose the underlying measure space $\{\Omega, \mu\}$ into a finite number of sets $\{E_1, \ldots, E_N\}$. They are the totality of the following sets:

$$\Omega[f \leqq g] \cap \Omega[k/2^m \leqq f < (k+1)/2^m] \cap \Omega[j/2^m \leqq g < (j+1)/2^m],$$

$$\Omega[f > g] \cap \Omega[k/2^m \leqq f < (k+1)/2^m] \cap \Omega[j/2^m \leqq g < (j+1)/2^m],$$

$$\Omega[f \leqq g] \cap \Omega[g \geqq m] \quad \text{and} \quad \Omega[f > g] \cap \Omega[f \geqq m],$$

where m is a positive integer and k, j run through all integers in the interval

$[0, m2^m)$. Using this decomposition, we define a quasi-norm preserving map: $h \to \bar{h} = (\bar{h}_1, \dots, \bar{h}_N)$ from T into l_N^s such that

$$\bar{h}_n = \left(\int_{E_n} h^s d\mu \right)^{1/s}, \qquad 1 \leqq n \leqq N.$$

Observe that for each n and nonnegative constants p, q, $p\bar{f}_n + q\bar{g}_n \leqq \overline{(pf + qg)}_n$, so that

$$\| p\bar{f} + q\bar{g} \|_s \leqq \| \overline{pf + qg} \|_s = \| pf + qg \|_s$$

by Minkowski's inequality. Now, by the very definition of the integral, the summation $\Sigma_{\mathrm{I}}(\bar{f}_n - \bar{g}_n)\bar{g}_n^{s-1}$ taken over all n's with $E_n \subset \Omega[f \leqq g]$ converges to $\int_{\Omega[f \leqq g]} (f-g) g^{s-1} d\mu$ as $m \to \infty$. Also by Fatou's lemma, the summation $\Sigma_{\mathrm{II}}(\bar{f}_n - \bar{g}_n)\bar{g}_n^{s-1}$ taken over all remaining n's satisfies

$$\underline{\lim}_m \sum_{\mathrm{II}} \geqq \int_{\Omega[f > g]} (f-g) g^{s-1} d\mu \qquad \left(\text{possibly } \sum_{\mathrm{II}} = +\infty \right).$$

So from the assumption, we are led to the inequality $\Sigma_1^N (\bar{f}_n - \bar{g}_n)\bar{g}_n^{s-1} > 0$ for sufficiently large m. Fix such an m, and pick up a small number $\varepsilon > 0$. Let us estimate the quasi-norm

$$\| \varepsilon \bar{f} + (1-\varepsilon)\bar{g} \|_s = \left\{ \sum_1^N [\bar{g}_n + \varepsilon(\bar{f}_n - \bar{g}_n)]^s \right\}^{1/s}.$$

Let $D_R \psi(\varepsilon)$ denote the right derivative of $\psi(\varepsilon)$. If either of the relations $\bar{g}_n > 0$ or $\bar{g}_n = \bar{f}_n = 0$ holds for each n, $1 \leqq n \leqq N$, then

$$D_R \| \varepsilon \bar{f} + (1-\varepsilon)\bar{g} \|_s^s \big|_{\varepsilon = 0} = s \sum_{n=1}^N \bar{g}_n^{s-1} (\bar{f}_n - \bar{g}_n) > 0.$$

If on the other hand there are n's such that $\bar{g}_n = 0$ and $\bar{f}_n > 0$, we get $D_R \| \varepsilon \bar{f} + (1-\varepsilon)\bar{g} \|_s^s \big|_{\varepsilon=0} = +\infty$. Therefore in both cases, for ε small enough we have $\| \bar{g} \|_s^s < \| \varepsilon \bar{f} + (1-\varepsilon)\bar{g} \|_s^s$, so that

$$\| g \|_s < \| \varepsilon \bar{f} + (1-\varepsilon)\bar{g} \|_s \leqq \| \overline{\varepsilon f + (1-\varepsilon) g} \|_s = \| \varepsilon f + (1-\varepsilon) g \|_s.$$

This contradicts the maximality of $\| g \|_s$.

THEOREM 2 (B. Maurey [1]). *Let $\{\Omega, \mu\}$ be a measure space and let E and G be two quasi-normed spaces. Furthermore let p, q, r be real numbers such that $0 < p < q \leqq +\infty$, $1/p = 1/q + 1/r$. For a continuous linear map u from E into $L^p(\Omega, \mu, G)$, the following conditions are equivalent*:

(1) *u is factored as $E \xrightarrow{v} L^q(\Omega, \mu, G) \xrightarrow{(h)} L^p(\Omega, \mu, G)$, where (h) denotes the*

multiplication operator by the function $h \in L^r(\Omega, \mu)$ *and the linear map* $v(e) = u(e)/h$ $(e \in E)$ *is continuous.*

(2) *There exists a constant C such that every finite sequence* $\{e_n\}$ *in E satisfies the inequality*

$$\left\{\int \left(\sum \|u(e_n)\|_G^q\right)^{p/q} d\mu\right\}^{1/p} \leqq C\left(\sum \|e_n\|^q\right)^{1/q}.$$

The best possible constant C_{pq} *of C in* (2) *equals the infimum of* $\|v\| \cdot \|h\|_r$ *taken over all factorizations in* (1). *Moreover, there exist factorizations* $u = (h) \circ v$ *attaining this infimum,* $\|v\| \cdot \|h\|_r = C_{pq}$, *and in this case the functions* $\|v\| \cdot |h|$ *of* $L^r(\Omega, \mu)$ *are uniquely determined.*

PROOF. Applying Hölder's inequality, we can easily establish the implication (1) $\Rightarrow$ (2) and the inequality $C_{pq} \leqq \|v\| \cdot \|h\|_r$ (cf. [1]).

Conversely assume that a nonzero linear map u satisfies (2). By scaling u, we may assume, without loss of generality, that $C_{pq} = 1$.

In case that $q = +\infty$, we repeat the original argument of [1]. Note that condition (2) in the case $q = +\infty$ can be written as

$$\left(\int [\sup\{\|u(e_n)\|\}]^p d\mu\right)^{1/p} \leqq \sup\{\|e_n\|\},$$

$\{e_n\}$ a finite sequence in E. This implies that the lattice theoretic supremum g of the family $\{\|u(e)\| \in L^p : e \in E,\ \|e\| \leqq 1\}$ belongs to $L^p(\Omega, \mu)$ and satisfies $\|g\|_p \leqq 1 = C_{pq}$. Call $v(e) = u(e)/g$. Then with $L^q = L^\infty(\Omega, \mu, G)$, $\|v(e)\|_{L^\infty} \leqq \|e\|$, or equivalently $\|v\| \leqq 1$. Therefore $u = (g) \circ v$ gives a desired factorization with the property $\|v\| \cdot \|g\|_p \leqq 1$, and so we have $\|v\| \cdot \|g\|_p = 1$. Now, take any factorization $u = (h) \circ w$ such that $\|w\| \cdot \|h\|_p = 1$. By scaling, we may assume that $\|w\| = 1$. Then $\|u(e)/h\|_{L^\infty} \leqq 1$ if $\|e\| \leqq 1$. So $\|g/h\|_\infty \leqq 1$, i.e. $g \leqq |h|$ a.e. μ, while $\|g\|_p = \|h\|_p = 1$. These imply that $g = |h|$ a.e. μ.

In the sequel, we shall assume that $0 < p < q < +\infty$. Put $s = p/q$, $0 < s < 1$. We first observe that for each finite sequence $\{e_n\}$ in E, the function $\Sigma \|u(e_n)\|^q$ belongs to $L^s(\Omega, \mu)$. Noting this fact, let us consider the set T' of functions in $L^s(\Omega, \mu)$ which can be written as

$$\sum \|u(e_n)\|^q, \qquad \{e_n\} \text{ a finite sequence in } E \text{ such that } \sum \|e_n\|^q \leqq 1.$$

Clearly T' is convex. On the other hand, since the inequality in (2) reads as

$$\left\{\int \left(\sum \|u(e_n)\|^q\right)^s d\mu\right\}^{1/s} \leqq \sum \|e_n\|^q,$$

T' is bounded in $L^s(\Omega, \mu)$. Consequently, by Lemma 1, the closure of T' in $L^s(\Omega, \mu)$ has a unique function g whose quasi-norm is maximum in the set. Under the present situation, this extremal function satisfies the inequality: with $f \in T'$

$$\| g \|_s = 1 \quad \text{and} \quad \int (f-g)g^{s-1}d\mu \leqq 0 \quad \text{or} \int fg^{s-1}d\mu \leqq \| g \|_s^s = 1.$$

Put $k = g^{1/(q+r)}$. Then from the identity $r/(q+r) = p/q = s$, it follows that $\| k \|_r = 1$. Define $w(e)$ by $w(e) = u(e)/k$ $(e \in E)$. We claim that w is a continuous linear map from E into $L^q(\Omega, \mu, G)$ with the norm $\| w \| \leqq 1$. Pick up an arbitrary vector $e \neq 0$ of E. Since T' contains the function $\| u(e) \| / \| e \|$, we have

$$1 \geqq \int [\| u(e) \| / \| e \|]^q g^{s-1} d\mu = \int [\| u(e) \| / \| e \|]^q k^{-q} d\mu,$$

i.e., $\| e \| \geqq \| w(e) \|_{L^q}$. This guarantees the above property of w. Therefore $u = (k) \circ w$ gives a factorization of u satisfying $\| w \| \cdot \| k \|_r \leqq 1$, so that $\| w \| \cdot \| k \|_r = 1$, because $\| w \| \cdot \| k \|_r \geqq 1$ as pointed earlier.

Finally take any factorization $u = (h) \circ v$ with $\| v \| \cdot \| h \|_r = 1$. By scaling, we may normalize v, h as $\| v \| = \| h \|_r = 1$. Since g is in the closure of T', there exists a sequence $\{g_n\}$ in T' which converges to g a.e. μ and in the s-th mean. From the inequality

$$\int \left(\sum \| u(e_n) \|^q / | h |^q \right) d\mu = \sum \| v(e_n) \|_{L^q}^q \leqq \sum \| e_n \|^q,$$

it follows that $\int g_n / | h |^q d\mu \leqq 1$. Hence by Fatou's lemma $\int g | h |^{-q} d\mu \leqq 1$. This yields, together with Hölder's inequality, that

$$1 = \int g^s d\mu = \int (g | h |^{-q})^s | h |^{qs} d\mu = \int (g | h |^{-q})^s (| h |^r)^{1-s} d\mu$$

$$\leqq \left(\int g | h |^{-q} d\mu \right)^s \left(\int | h |^r d\mu \right)^{1-s} \leqq 1.$$

Since every term in the above is actually identical, we conclude that

$$g | h |^{-q} / | h |^r = \text{const.} \quad \text{a.e.} \quad \text{or} \quad k / | h | = \text{const.} \quad \text{a.e. } \mu.$$

This implies that $k = | h |$ a.e. μ, because $\| k \|_r = \| h \|_r = 1$.

THEOREM 3 (B. Maurey [1]). *Let $\{\Omega, \mu\}$ be a measure space and let E and g be two quasi-normed spaces. Furthermore let p, q, r be real numbers such that*

$0 < p < q < +\infty$, $1/p = 1/q + 1/r$. *Suppose u is a continuous linear map from a linear subspace S_q of $L^q(\Omega, \mu, G)$ into E. Then the following are equivalent:*

(1) *u is factored as $S_q \xrightarrow{(h)} S_p \xrightarrow{v} E$ with some $h \in L^r(\Omega, \mu)$ and v a continuous linear map.*

(2) *There exists a constant C such that every finite sequence $\{f_n\}$ in S_q satisfies the inequality*

$$\left\{\sum \| u(f_n)\|^p\right\}^{1/p} \leqq C\left\{\int \left(\sum \| f_n \|_G^p\right)^{q/p} d\mu\right\}^{1/q}.$$

The best possible constant C_{pq} of C in (2) *equals the infimum of $\| v \| \cdot \| h \|_r$ taken over all factorizations in* (1). *Moreover, there exist factorizations $u = (h) \circ v$ attaining this infimum, $\| v \| \cdot \| h \|_r = C_{pq}$, and in this case the functions $\| v \| \cdot | h |$ of $L^r(\Omega, \mu)$ are uniquely determined.*

REMARK. The following proof was suggested by the referee.

PROOF. The implication (1) $\Rightarrow$ (2) and the inequality $C_{pq} \leqq \| v \| \cdot \| h \|_r$ are immediate from Hölder's inequality (cf. [1]).

Conversely assume that a nonzero linear map u satisfies (2). By scaling u, we may assume that $C_{pq} = 1$. Put $s = q/p$, $1 < s < +\infty$, and consider the subset T' of $L^s(\Omega, \mu)$, each function of which is in the form

$$\sum \| f_n \|^p, \qquad \{f_n\} \text{ a finite sequence in } S_q \text{ such that } \sum \| u(f_n)\|^p = 1.$$

Clearly T' is convex and so its closure admits a unique function g with the minimum norm. Note that the inequality in (2) reads as

$$\left\{\int \left(\sum \| f_n \|^p\right)^s d\mu\right\}^{1/s} \geqq \sum \| u(f_n)\|^p.$$

Therefore $\| g \|_s = 1$, since $C_{pq} = 1$. Furthermore by Lemma 1, this function g satisfies

$$\int f g^{s-1} d\mu \geqq \| g \|_s^s = 1, \qquad \forall f \in T'.$$

So, if we put $k = g^{1/(r-p)}$, the above inequality can be expressed in the form

$$\int \| f \|^p g^{s-1} d\mu = \int \| f \|^p k^p d\mu \geqq \| u(f)\|^p, \qquad \forall f \in S_q. \tag{1.1}$$

Put $S_p = \{kf : f \in S_q\}$. It is easy to check, invoking Hölder's inequality that S_p is a linear subspace of $L^p(\Omega, \mu, G)$. Here we require that the correspondence

$w: S_p \ni kf \to u(f) \in E$ $(f \in S_q)$ gives a continuous linear map from S_p into E. Indeed, for any pair $f_1, f_2 \in S_q$, replacing f with $f_1 - f_2$ in (1.1), we find that $u(f_1) = u(f_2)$ whenever $kf_1 = kf_2$, so that w is well-defined. Again by (1.1), we get $\| w \| \leqq 1$. Thus the desired factorization $u = w \circ (k)$ has been obtained: clearly $k \in L^r(\Omega, \mu)$ and $\| k \|_r = 1$. Also the pair satisfy $\| w \| \cdot \| k \|_r \leqq 1$, and so $\| w \| \cdot \| k \|_r = 1$.

Finally take any factorization $u = v \circ (h)$ with the equality $\| v \| \cdot \| h \|_r = 1$. By scaling we may normalize h as $\| h \|_r = 1$. Since g is in the closure of T', there exists a sequence $\{g_n\}$ in T' which converges to g in the s-th mean. From the inequality

$$\sum \| u(f_n) \|^p = \sum \| v(hf_n) \|^p \leqq \int \left(\sum \| hf_n \|^p \right) d\mu,$$

it follows that $\int g_n \, | h |^p d\mu \geqq 1$. Therefore, using Hölder's inequality, we get $\int g \, | h |^p d\mu \geqq 1$, and so

$$1 \leqq \int g \, | h |^p d\mu \leqq \left\{ \int g^s d\mu \right\}^{1/s} \left\{ \int | h |^r d\mu \right\}^{p/r} = 1.$$

Since every term in the above is identical, we conclude that $g^s / | h |^r = k^r / | h |^r =$ const. a.e. μ, so that $k = | h |$ a.e. μ, because $\| k \|_r = \| h \|_r = 1$.

Reference

1. B. Maurey, *Théorème de factorisation pour les opérateurs linéaires à valeurs dans les espaces* L^p, Astérisque **11** (1974).